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On the Penning trap coherent states

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Abstract

Recently, a class of coherent states of a particle in a Penning trap was derived by Fernández and Velázquez (2009 *J. Phys. A: Math. Theor.* **42** 085304). By means of the Wigner function and density matrix associated with these states, we show that they are fully consistent with Morikawa's definition of the decoherence degree and hence they provide a possibility to directly access the decoherence process in a Penning trap.

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1. Introduction

Quantum decoherence has been intensively studied in the last decades, since it is crucial for a fundamental understanding of quantum mechanics and also gains increasing practical relevance in potential quantum computer implementations. Briefly, the decoherence process can be defined as entanglement formation of a quantum system with its environment. If the system is initially prepared in a coherent state at $t = 0$ and exposed to some kind of interaction with its environment, the quantum coherence decreases with time, i.e. the properties of a coherent state are no longer fulfilled for $t > 0$. One of the most convenient approaches to investigate the decoherence behavior of a quantum system are density matrices, since it is possible to define a dimensionless decoherence degree in terms of the so-called 'coherence length' and 'ensemble width' which, under certain conditions, can be directly extracted from the coordinate representation of the density matrix.

The purpose of the present paper is to demonstrate that the recently derived Penning trap coherent states [1] indeed maximize the above-mentioned coherence definition (which is not obvious from their purely algebraic derivation) and are therefore well suited for decoherence studies in Penning traps. We proceed as follows: in section 2, we briefly recall the definition and necessary properties of these particular coherent states as derived in [1]. In section 3 we give an analytical expression for the Wigner function and the density matrix associated with the Penning trap coherent states in coordinate representation. Morikawa's definition of the dimensionless decoherence degree is introduced in section 4, and the main result is presented and discussed subsequently. In particular, we show that the consistency of the Penning trap coherent states with the decoherence degree definition is a direct consequence of the fact that (i) the covariance matrix of these states is diagonal and (ii) that these states minimize the Heisenberg uncertainty relation as shown in [1].

2. Penning trap coherent states

Coherent states [2] are an important tool in quantum mechanics and have also been studied in connection with ion traps, e.g. in a Paul trap [3], and, most recently, in a Penning trap [1], where the following Hamiltonian of a spinless particle with charge e (here and in the following we set the mass $m = 1$ and $\hbar = 1$) was considered

$$H = \frac{\mathbf{p}^2}{2} + bL_z + \frac{1}{2}[(b^2 + \nu)(x^2 + y^2) - 2\nu z^2]. \tag{1}$$

Here, $b = -eB/(2c) > 0$ (c being the speed of light), $\nu = 2e\Phi_0/d^2 < 0$, B is a constant magnetic field pointing in the z-direction, d the characteristic trap dimension and Φ_0 the strength of the electrostatic potential Φ

$$\Phi(\mathbf{r}) = \frac{\Phi_0}{d^2}(x^2 + y^2 - 2z^2). \tag{2}$$

In addition, the constants b and ν satisfy the relation $b^2 + \nu > 0$. The corresponding coherent states $|z_1, z_2, z_3\rangle = |\mathbf{z}\rangle$ can be obtained in coordinate representation starting from the extremal state $|0, 0, 0\rangle$,

$$\langle \mathbf{r}|0, 0, 0\rangle = \varphi_{\mathbf{0}}(\mathbf{r}) = \alpha \exp\left(-\frac{\sqrt{b^2 + \nu}}{2}(x^2 + y^2) - \sqrt{\frac{-\nu}{2}}z^2\right), \tag{3}$$

(where α is a normalization constant), in the following way:

$$\varphi_{\mathbf{z}}(\mathbf{r}) = C(\mathbf{z})F(\mathbf{r})\varphi_{\mathbf{0}}\left(x - \frac{\mathrm{Re}[z_1 - z_2]}{(b^2 + \nu)^{\frac{1}{4}}}, y + \frac{\mathrm{Im}[z_1 + z_2]}{(b^2 + \nu)^{\frac{1}{4}}}, z + \left(\frac{-2}{\nu}\right)^{\frac{1}{4}} \mathrm{Im}[z_3]\right), \tag{4}$$

with

$$C(\mathbf{z}) = \exp(\mathrm{i}(\mathrm{Re}[z_1]\,\mathrm{Im}[z_2] + \mathrm{Re}[z_2]\,\mathrm{Im}[z_1] + \mathrm{Re}[z_3]\,\mathrm{Im}[z_3])), \tag{5}$$

$$F(\mathbf{r}) = \exp\big(\mathrm{i}(b^2 + \nu)^{\frac{1}{4}}(\mathrm{Im}[z_1 - z_2]x + \mathrm{Re}[z_1 + z_2]y) + \mathrm{i}(-8\nu)^{\frac{1}{4}}\,\mathrm{Re}[z_3]z\big). \tag{6}$$

For the following calculations, we also require the expectation values and covariances corresponding to the states (4). We denote the vector of expectation values by

$$\mathbf{m} = (\langle x\rangle, \langle y\rangle, \langle z\rangle, \langle p_x\rangle, \langle p_y\rangle, \langle p_z\rangle)^T \tag{7}$$

and define the covariances of two operators A, B

$$\sigma_{AB} = \tfrac{1}{2}\langle AB + BA\rangle - \langle A\rangle\langle B\rangle, \tag{8}$$

which determine the symmetric covariance matrix σ. We adopt the notation $\sigma_{ij} = \sigma_{ji}$ where $i, j = 1, 2, 3$ correspond to the coordinates x, y, z and $i, j = 4, 5, 6$ to the momenta p_x, p_y, p_z. The expectation values were already calculated in [1]

$$\begin{aligned}
&\langle x\rangle = (b^2 + \nu)^{-\frac{1}{4}}\,\mathrm{Re}[z_1 - z_2], \qquad \langle y\rangle = -(b^2 + \nu)^{-\frac{1}{4}}\,\mathrm{Im}[z_1 + z_2],\\
&\langle z\rangle = -(-\nu/2)^{-\frac{1}{4}}\,\mathrm{Im}[z_3],\\
&\langle p_x\rangle = (b^2 + \nu)^{\frac{1}{4}}\,\mathrm{Im}[z_1 - z_2], \qquad \langle p_y\rangle = (b^2 + \nu)^{\frac{1}{4}}\,\mathrm{Re}[z_1 + z_2],\\
&\langle p_z\rangle = (-8\nu)^{\frac{1}{4}}\,\mathrm{Re}[z_3].
\end{aligned} \tag{9}$$

As for the covariances, [1] gives the diagonal values of σ,

$$\sigma_{11} = (4(b^2+\nu))^{-\frac{1}{2}} = \sigma_{22}, \qquad \sigma_{33} = (-8\nu)^{-\frac{1}{2}},$$
$$\sigma_{44} = \tfrac{1}{2}\sqrt{b^2+\nu} = \sigma_{55}, \qquad \sigma_{66} = \sqrt{-\nu/2}, \tag{10}$$

and it is straight forward to obtain the remaining off-diagonal elements. First, we note that the states (4) can be factorized as product states, $\varphi_{\mathbf{z}}(\mathbf{r}) = f(x)g(y)h(z)$, and hence it follows directly from the definition (8) that all the covariances of the form $\sigma_{r_i r_j}$, $\sigma_{p_i p_j}$ and $\sigma_{r_i p_j}$ vanish for $i \neq j$. Therefore, one only needs to compute $\sigma_{xp_x} = \sigma_{14}$, $\sigma_{yp_y} = \sigma_{25}$, $\sigma_{zp_z} = \sigma_{36}$, which can be done analytically and gives

$$\sigma_{14} = \sigma_{25} = \sigma_{36} = 0. \tag{11}$$

We see that the covariance matrix of the Penning trap coherent states is diagonal, with the diagonal elements given by (10). In addition, the generalized uncertainty relation is minimal for each mode

$$\sigma_{r_i r_i}\sigma_{p_i p_i} - \sigma_{r_i p_i}^2 = \sigma_{r_i r_i}\sigma_{p_i p_i} = \tfrac{1}{4}, \qquad i = 1, 2, 3. \tag{12}$$

In the following section, we use these results to calculate the Wigner function and the density matrix in coordinate representation.

3. Wigner function and density matrix

It is well known [4–8] that for a Hamiltonian which is quadratic in the canonical variables and an initial state of Gaussian type, the corresponding Wigner function f_W remains Gaussian for all times and hence the dynamics is fully determined by the time evolution of the expectation values and covariances

$$f_W(\mathbf{r}, \mathbf{p}, t) = \frac{1}{\sqrt{\det(2\pi\sigma(t))}} \exp\left(-\frac{1}{2}(\xi - \mathbf{m}(t))^{\mathrm{T}}\sigma(t)^{-1}(\xi - \mathbf{m}(t))\right), \tag{13}$$

where $\xi = (x, y, z, p_x, p_y, p_z)^{\mathrm{T}}$ is the phase space vector. The density matrix at any time t can then be obtained in coordinate representation by the transformation

$$\langle \mathbf{r}|\rho|\mathbf{r}'\rangle(t) = \int \mathrm{d}^3\mathbf{p} \exp(\mathrm{i}(\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')) f_W((\mathbf{r}+\mathbf{r}')/2, \mathbf{p}, t). \tag{14}$$

In particular, if the initial state is a Penning trap coherent state (4), we can insert the previous results into this expression so that the Wigner function becomes

$$f_W((\mathbf{r}+\mathbf{r}')/2, \mathbf{p}, t=0) = \frac{1}{\pi^3} \exp\left(2\sqrt{b^2+\nu}(\bar{x}^2+\bar{y}^2) + \sqrt{-8\nu}\bar{z}^2 + \frac{2}{\sqrt{b^2+\nu}}\left(\bar{p_x}^2 + \bar{p_y}^2\right) + \sqrt{-2/\nu}\,\bar{p_z}^2\right), \tag{15}$$

where we used the abbreviations

$$\bar{r}_i = \frac{r_i + r_i'}{2} - \langle r_i\rangle, \qquad \bar{p}_i = p_i - \langle p_i\rangle. \tag{16}$$

After evaluating the integral in (14), which can be done analytically, we obtain the following expression for the density matrix:

$$\langle \mathbf{r}|\rho|\mathbf{r}'\rangle(t=0) = \left(\frac{2}{\pi}\right)^{3/2}\left(\frac{4}{b^2+\nu}\right)^{-1/2}\left(-\frac{2}{\nu}\right)^{-1/4} \exp(\mathrm{i}(\langle p_x\rangle\Delta_x + \langle p_y\rangle\Delta_y + \langle p_z\rangle\Delta_z)$$
$$\times \exp\left(-\frac{1}{2}\left(2\sqrt{b^2+\nu}(\bar{x}^2+\bar{y}^2) + \sqrt{-8\nu}\bar{z}^2 + \frac{\sqrt{b^2+\nu}}{2}\left(\Delta_x^2+\Delta_y^2\right) + \sqrt{-\nu/2}\Delta_z^2\right)\right), \tag{17}$$

where

$$\Delta_{r_i} = r_i - r_i'. \tag{18}$$

4. Decoherence degree, main result and conclusion

To the best of our knowledge, a dimensionless decoherence degree for Gaussian-type density matrices was for the first time suggested by Morikawa [9]. It was adopted in later works [10–12] and can also be found in recent textbooks [13, 14]. If the density matrix has the form

$$\rho(x, x', t) = \mathcal{N}(t) \exp\left(-A(t)(x - x')^2 - \mathrm{i}B(t)(x - x')(x + x') - C(t)\left(\frac{x + x'}{2}\right)^2\right), \tag{19}$$

then quantum decoherence can be described by the amplitude of the off-diagonal elements, or, as sometimes referred to in literature, by the characteristic coherence length $l(t) = 1/\sqrt{8A(t)}$ and ensemble width $\Delta X(t) = 1/\sqrt{2C(t)}$. These two quantities allow a definition of a dimensionless degree of quantum decoherence

$$\delta_{QD}(t) = \frac{l(t)}{\Delta X(t)} = \frac{1}{2}\sqrt{\frac{C(t)}{A(t)}}, \tag{20}$$

such that $\delta_{QD} = 0$ corresponds to a total loss of quantum coherence. This definition is derived from the position representation of the density operator, and it is therefore sometimes also referred to as spatial decoherence in the literature. The physical reasons for this choice are discussed in detail in [14] (see, e.g., introduction to chapter 3 therein), and here we only mention one of the main arguments, namely that probably the most important decoherence source is given by scattering processes between the reduced system and environmental particles. This leads to '*. . . system-environment entanglement that delocalizes local phase relations between spatially separated wave-function components, leading to decoherence in position space (i.e.,to localization)*'. Technically, however, it is as well possible to define a decoherence degree in momentum space, at least for the particular case studied in the present paper. The position representation of the density operator from which the decoherence degree (20) was extracted is obtained by integrating out the momenta in the Wigner function (cf equation (14)), but one can equally treat the dynamics in momentum space. Thus, the momentum representation of the density operator is obtained by integrating out the coordinates in the Wigner function and, since the Hamiltonian is quadratic in the canonical variables, the transformation yields the same Gaussian structure as in (19), where x, x' are replaced by p, p' and the coefficients $A(t)$, $B(t)$ and $C(t)$ have a different form. The degree of momentum decoherence can then, in principle, be defined in the same manner, but the coefficients $A(t)$ and $C(t)$ can no longer be interpreted in terms of a coherence length and an ensemble width, at least not in the common sense. However, in this context we would like to mention that, although the vast majority of master equations used to describe collisional decoherence [15–19] deals with the time evolution of the density operator in position representation $\rho(\mathbf{r}, \mathbf{r}', t)$, a possibility of extracting information about decoherence from its momentum representation $\rho(\mathbf{p}, \mathbf{p}', t)$ was indeed presented recently [20]. Therein, the author also points out that the answer to the question whether momentum decoherence occurs in addition to spatial decoherence or not is directly related to the mass ratio of the tracer and the particles in the environment, that is, if this ratio is negligible, so is momentum decoherence. The physical interpretation of momentum decoherence is then different from the one quoted previously, since it does not describe spatial localization but is related to the rate of particles being scattered off the environment into different directions (or with different velocities).

Combining the result from the previous section (17) with the decoherence degree definition (20), it follows that the Penning trap coherent states satisfy $\delta_{QD}(t=0)=1$ for each mode and are therefore perfectly coherent in line with Morikawa's definition

$$\delta^x_{QD}(t=0)=\delta^y_{QD}(t=0)=\frac{1}{2}\sqrt{4\frac{\sqrt{b^2+\nu}}{\sqrt{b^2+\nu}}}=1,\qquad \delta^z_{QD}(t=0)=\frac{1}{2}\sqrt{\frac{\sqrt{-8\nu}}{\sqrt{-\nu/2}}}=1. \tag{21}$$

As can be seen, this is a direct consequence of their property to minimize the uncertainty relation (12). This result allows future studies of decoherence processes in Penning traps: as previously mentioned, the dynamics of the system (and, in particular, the decoherence degree itself) is fully given by the time evolution of the first two moments. If the latter one is obtained from unitary phase space dynamics (e.g. time-dependent Schrödinger equation [21, 22] or by means of the Weyl–Wigner–Moyal propagator [23]), quantum coherence ($\delta_{QD}=1$) is preserved for all times. However, if instead of a unitary time evolution the expectation values and variances are obtained within a model that takes into account environmental effects, one also directly obtains the decoherence degree of the system as a function of time, and the coherent states (4) are, as just shown, excellent candidates for an initial state. As an example, we mention the Markovian master equation based on semigroups [24], where decoherence is described by Lindblad operators [25–27]. This makes it possible to study the dependence of the decoherence rate on parameters like temperature or environmental coupling strength. Investigations in this direction could be relevant in view of possible quantum computational applications of electron or ion traps [28–40]. Note, however, that in a full time-dependent treatment the covariance matrix is not necessarily diagonal at all times, which makes the general time-dependent expressions for the coherence length and the ensemble width (and hence the decoherence degree) more complicated than the initial ones explicitly given here.

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